Letters in Mathematical Physics (2021) 111:103
https://doi.org/10.1007/s11005-021-01443-9

Determination of the rich structural wave dynamic solutions to the Caudrey–Dodd–Gibbon equation and the Lax equation

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Received: 5 July 2018 / Revised: 18 May 2021 / Accepted: 11 July 2021

Abstract

This article addresses the implementation of the new generalized (G'/G)-expansion method to the Caudrey–Dodd–Gibbon (CDG) equation and the Lax equation which are associated with the fifth-order KdV (fKdV) equation. The method works well to derive a variety of standard and functional closed-form wave solutions with distinct physical structures, such as, soliton, kink, periodic soliton, and bell-shaped soliton solutions. The solutions obtained using this method are useful and adequate than other methods. In order to understand the physical aspects and importance of the method, the attained solutions have been simulated graphically. The extracted results definitely establish that the new generalized (G'/G)-expansion method is an effective mathematical tool to work out new solutions to different types of local nonlinear evolution equations emerging in applied science and engineering, but this method is not effective in solving nonlocal equations.

Keywords Caudrey–Dodd–Gibbon (CDG) equation · Lax equation · Soliton solution · Nonlinear evolution equations · New generalized (G'/G)-expansion method

Mathematics Subject Classification 35C08 · 47J35 · 35R35

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Published online: 29 July 2021

1 Introduction

It is well established that almost every instinctual phenomenon is nonlinear and mathematically appears in the form of nonlinear evolution equations (NLEEs). The studies of NLEEs, a special type of nonlinear partial differential equations (NPDEs), become one of the most exciting and highly active areas of research and investigation, because problems in various scientific and engineering fields, such as, solid state physics, chemical physics, plasma physics, optics, biology, chemical kinematics, geochemistry, fluid mechanics and hydrodynamics, are frequently described by NLEEs. In order to understand the inner structure of these phenomena, finding closed-form soliton solutions is becoming more fascinating day-by-day. But there is no integrated method which could be utilized to deal with all types of NLEEs. That is why a variety of efficient and reliable methods have been developed, videlicet, the Painleve expansion method [1], the inverse scattering method [2, 3], the Darboux transformation method [4, 5], the Cole–Hopf transformation method [6, 7] the Jacobi elliptic function method [8, 9], the Hirota's bilinear transformation method [10, 11], the Backlund transformation method [12, 13], the sine–cosine method [14], the tanh function method [15–17], the F-expansion method [18], the Kudryashov [19], the exp-function method [20], the $\exp(-\phi(\xi))$-expansion method [21], the modified simple equation method [22], the $\left(G'/G\right)$-expansion method [23–26], the new generalized $\left(G'/G\right)$-expansion method [27, 28], the double $\left(G'/G, 1/G\right)$- expansion method [29], etc.

This article is concerned with the Caudrey–Dodd–Gibbon (CDG) equation and the Lax equation that are used to model nonlinear dispersive waves in diverse scientific fields such as laser optics, plasma physics. The CDG equation is integrable nonlinear fifth-order equation and is of the form [30, 31]:

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0 \tag{1}$$

The fifth-order Lax equation is also nonlinear integrable equation and is of the form [32]:

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = 0 \tag{2}$$

These two equations have been shown to be related to the integrable cases of the Henon–Heiles [33] system. However, Eqs. (1) and (2) have been studied successively in a series of articles [17, 34–47]. Wazwaz investigated different types of solutions of Eqs. (1) and (2) by using several methods, namely, the tanh function method [17, 34, 35], the sine–cosine method [35], the extended tanh method [36], the tanh–coth method [37], the Hirota's bilinear method [38, 39], etc. The obtained solutions include periodic, soliton and multiple soliton, etc. Moreover, closed-form solitary wave solutions to the CDG equation and the Lax equation were derived by Bilige and Chaolu by using the extended simplest equation method [40]. Furthermore, solutions of Eq. (1) examined by Salas [41] by using the projective Riccati equation method, Xu et al. [42] employed the exp-function method, Gomez and Salas [43] utilized the generalized tanh-coth method, Jin [44] applied the variational iteration method, Naher et al. [24] implemented the $\left(G'/G\right)$-expansion method and Bisaws et al. [45] used the modified F-expansion method, exp-function method as well as the $\left(G'/G\right)$ method. Also, solutions of Eq. (2) investigated by Abbasbandy and Zakaria [46] by means of the homotopy analysis method and Gomez [47] used the generalized extended tanh method. However, no one studied the solutions to the aforesaid equations through the new generalized $\left(G'/G\right)$-expansion method. In this article, our aim is to investigate Eqs. (1) and (2) by using the new generalized $\left(G'/G\right)$-expansion method and establish further closed-form solitary wave solutions which include singular soliton, kink, singular kink, bell-shaped soliton, anti-bell-shaped soliton, periodic, and bell-type solitary wave solutions.

This article is organized as follows: In Sect. 2, we have reviewed briefly the new generalized $\left(G'/G\right)$-expansion method. In Sect. 3, we have presented the application of the method to Eqs. (1) and (2) to extracted abundant closed-form solitary wave solutions. In Sect. 4, we have provided the physical explanation and graphical presentation of the obtained solutions. Finally, in Sect. 5, conclusions are drawn.

2 The new generalized $\left(G'/G\right)$-expansion method

Let us consider a general NLEE in the form

$$P\left(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \dots\right) = 0 \tag{3}$$

where $u = u(x, t)$ is an unspecified function, P is a polynomial in $u(x, t)$ and its partial derivatives wherein the highest order derivatives and the nonlinear terms are associated. The key steps of the new generalized (G'/G)-expansion method are given in the succeeding:

Step 1 We suppose that the combination of spatial variables x and the temporal t by a compound variable ξ as follows:

$$u(x, t) = u(\xi),\ \ \xi = x - ct \tag{4}$$

where c is the speed of the solitary wave. The wave transformation (4) allows us to moderate Eq. (3) to an ordinary differential equation (ODE) for $u = u(\xi)$ in the form:

$$R\left(u, u', u'', u''', \dots\right) = 0 \tag{5}$$

wherein R is a function of $u(\xi)$ and the superscripts indicate the ordinary derivatives regarding to ξ.

Step 2 In particular case, sometimes Eq. (5) can be integrated term by term one or more times, yields zero integral constant(s).

Step 3 We assume that the closed-form solitary wave solution of (5) can be revealed as follows, in accordance with the new generalized $\left(G'/G\right)$-expand method:

$$u(\xi) = \sum_{k=0}^{N} a_k(d + Y(\xi))^k + \sum_{k=1}^{N} b_k(d + Y(\xi))^{-k} \tag{6}$$

in which either a_N or b_N may be zero, but both a_N and b_N cannot be zero at a time, a_k, $b_k(k = 0, 1, 2, 3, \ldots N)$ and d are indefinite constants to be calculated afterword and $Y(\xi)$ is given by

$$Y(\xi) = \left(G'/G\right), \tag{7}$$

where in $G = G(\xi)$ satisfies the next auxiliary nonlinear differential equation

$$AGG'' - BGG' - EG^2 - C\left(G'\right)^2 = 0 \tag{8}$$

where prime specifies the derivative with respect to ξ and A, B, C, E are real parameters.

Step 4 The number of terms in (6) will be fixed by the value of N, and its value will be determined by the homogeneous balancing principle.

Step 5 Inserting solution Eq. (6) along with (8) including (7) into (5) in conjunction with the value of N attained in Step 4, we reach a polynomial in $(d + Y)^N$, and $(d + Y)^{-N}$, $(N = 1, 2, \ldots)$. We set each coefficient of the resulting polynomial to zero, yield an over-determined set of algebraic equations for a_k, $b_k(k = 1, 2, \ldots, N)$, d and c.

Step 6: We stare that the value of the constants can be determined by solving the algebraic equations achieved in Step 5. Since the general solution of (8) is in general known, inserting the value of $a_k(k = 0, 1, 2, \ldots, N)$, $b_k(k = 1, 2, \ldots, N)$, d and c into solution (6) yields the comprehensive and newly produced exact traveling wave solutions to the nonlinear evolution Eq. (3).

Step 7 By means of the general solution of Eq. (8), we admits the following solution of Eq. (7).

Family 1: When $B = 0$, $\psi = A - C$ and $\Omega = E\psi > 0$,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{\Omega}}{\psi} \frac{r \sinh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right) + s \cosh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)}{r \cosh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right) + s \sinh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)} \tag{9}$$

Family 2: When $B = 0$, $\psi = A - C$ and $\Omega = E\psi < 0$,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Omega}}{\psi} \frac{-r \sin(\frac{\sqrt{-\Omega}}{\psi}\xi) + s \cos(\frac{\sqrt{-\Omega}}{\psi}\xi)}{r \cos(\frac{\sqrt{-\Omega}}{\psi}\xi) + s \sin\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)} \tag{10}$$

Family 3: When $B \neq 0$, $\psi = A - C$ and $\Delta = B^2 + 4E(A - C) > 0$,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi} \frac{r \sinh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right) + s \cosh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)}{r \cosh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right) + s \sinh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)} \tag{11}$$

Family 4: When $B \neq 0$,
$\psi = A - C$ and $\Delta = B^2 + 4E(A - C) < 0$,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\frac{-r\sin(\frac{\sqrt{-\Delta}}{2\psi}\xi) + s\cos(\frac{\sqrt{-\Delta}}{2\psi}\xi)}{r\cos(\frac{\sqrt{-\Delta}}{2\psi}\xi) + s\sin\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)} \tag{12}$$

Family 5: When $B \neq 0$, $\psi = A - C$ and $\Delta = B^2 + 4E(A - C) = 0$,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{s}{r + s\xi} \tag{13}$$

3 Formulation of the solutions

In this section, we will analyze two NLEEs, namely, the Caudrey–Dodd–Gibbon (CDG) equation and the Lax equation and establish useful solutions by using the new generalized $\left(G'/G\right)$-expansion method described in Sect. 2.

3.1 The Caudrey–Dodd–Gibbon equation

In this sub-section, we will study the nonlinear CDG Eq. (1). The CDG equation can be written in the form [30, 31]:

$$u_t + \frac{\partial}{\partial x}\left(u_{xxxx} + 30uu_{xx} + 60u^3\right) = 0 \tag{14}$$

My means of the wave transformation $u(x,t) = u(\xi)$, $\xi = x - ct$, Eq. (14) reduces to the ordinary differential equation

$$-cu' + \left(u^{(iv)} + 30uu'' + 60u^3\right)' = 0 \tag{15}$$

On integrating (15) with respect to ξ once and letting the constant of integration to zero, we obtain

$$-cu + u^{(iv)} + 30uu'' + 60u^3 = 0 \tag{16}$$

According to the method described in Sect. 2, and after balancing, we obtain $N=2$. Therefore, the solution of (16) turns into the form

$$u(\xi) = a_0 + a_1(d + Y) + a_2(d + Y)^2 + b_1(d + Y)^{-1} + b_2(d + Y)^{-2} \tag{17}$$

where a_0, a_1, a_2, b_1, b_2 and d are constants to be determined later on.

Now introducing (17) into (16) including (7) and (8), the left hand side of Eq. (16) is translated into the polynomial in $(d + Y)^N$ and $(d + Y)^{-N}$, $(N = 1, 2, \ldots)$. Equalizing the cohorts of this polynomial to zero, we obtain an algebraic system (for simplicity, we leave out in displaying the equations) with respect to $a_0, a_1, a_2, b_1, b_2, c$ and d.

Solving the system of algebraic equations with the aid of the Maple 17, we obtain the following sets of distinct solutions of parameters $a_0, a_1, a_2, b_1, b_2, c$ and d:

Case 1

$$a_0 = -\frac{d^2\psi^2 + (Bd - E)\psi}{A^2},\ a_1 = \frac{2d\psi^2 + B\psi}{A^2},\ a_2 = -\frac{\psi^2}{A^2},\ b_1 = 0,\ b_2 = 0,\ c = \frac{\left(B^2 + 4\Omega\right)^2}{A^4} \tag{18}$$

where $\psi = A - C, \Omega = E\psi, d, A, B, C$ and E are free parameters.
Case 2

$$\begin{aligned} a_0 &= -\frac{d^2\psi^2 + (Bd - E)\psi}{A^2},\ a_1 = 0,\ a_2 = 0, \\ b_1 &= \frac{d\psi\left(2d^2\psi + 3Bd - 2E\right) + B(Bd - E)}{A^2}, \\ c &= \frac{\left(B^2 + 4\Omega\right)^2}{A^2},\ b_2 = -\frac{d\psi\left(d^3\psi + 2d^2B - 2dE\right) + (Bd - E)^2}{A^2} \end{aligned} \tag{19}$$

where $\psi = A - C, \Omega = E\psi, d, A, B, C$ and E are free parameters.
Case 3

$$\begin{aligned} a_0 &= \frac{-d\psi(d\psi + B) + \frac{1}{2}\left(1 \pm \frac{\sqrt{105}}{15}\right)\Omega - \frac{1}{8}\left(1 \pm \frac{\sqrt{105}}{15}\right)B^2}{A^2},\ a_1 = \frac{2d\psi^2 + B\psi}{A^2}, \\ a_2 &= -\frac{\psi^2}{A^2},\ b_1 = 0,\ b_2 = 0,\ c = \frac{(11 \pm \sqrt{105})\left(B^2 + 4\Omega\right)^2}{8A^4} \end{aligned} \tag{20}$$

where $\psi = A - C,\Omega = E\psi, d, A, B, C$ and E are free parameters.
Case 4

$$\begin{aligned} a_0 &= \frac{-d\psi(d\psi + B) + \frac{1}{2}\left(1 \pm \frac{\sqrt{105}}{15}\right)\Omega - \frac{1}{8}\left(1 \pm \frac{\sqrt{105}}{15}\right)B^2}{A^2},\ a_1 = 0,\ a_2 = 0 \\ b_1 &= \frac{d\psi\left(2d^2\psi + 3Bd - 2E\right) + B(Bd - E)}{A^2} \end{aligned} \tag{21}$$

where $\psi = A - C,\Omega = E\psi, d, A, B, C$ and E are free parameters.
Case 5

$$\begin{aligned} a_0 &= \frac{\left(B^2 + 4\Omega\right)}{2A^2},\ a_1 = 0,\ a_2 = -\frac{\psi^2}{A^2}, \\ b_1 &= 0,\ b_2 = -\frac{\left(B^2 + 4\Omega\right)^2}{16A^2\psi^2},\ c = \frac{16\left(B^2 + 4\Omega\right)^2}{A^4},\ d = -\frac{B}{2\psi} \end{aligned} \tag{22}$$

where $\psi = A - C,\Omega = E\psi, A, B, C$ and E are free parameters.
Case 6

$$a_0 = \frac{\sqrt{105}\left(B^2+4\Omega\right)}{30A^2},\ a_1 = b_1 = 0,\ a_2 = -\frac{\psi^2}{A^2},\ b_2 = -\frac{\left(B^2+4\Omega\right)^2}{16A^2\psi^2}$$
$$c = \frac{2(11 \pm \sqrt{105})\left(B^2+4\Omega\right)^2}{A^4},\ d = -\frac{B}{2\psi} \tag{23}$$

where $\psi = A - C, \Omega = E\psi$, A, B, C and E are free parameters.

For Case 1

Inserting the values of the parameters assembled in (18) into solution (17) and combining the solutions (9) to (13) and simplifying, we attain the following closed-form solitary wave solutions for $r = 0$ but $s \neq 0$, respectively.

$$u_{11}(\xi) = \alpha_1 + \alpha_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$$

$$u_{12}(\xi) = \alpha_1 + \alpha_2\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$$

$$u_{13}(\xi) = \alpha_1 + \alpha_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$$

$$u_{14}(\xi) = \alpha_1 + \alpha_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2,$$

$$u_{15}(\xi) = \alpha_1 + \alpha_2\left[d + \frac{B}{2\psi} + \frac{1}{\xi}\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{1}{\xi}\right]^2$$

where $\alpha_1 = -\frac{d^2\psi^2+(Bd-E)\psi}{A^2}, \alpha_2 = \frac{2d\psi^2+B\psi}{A^2}$ and $\xi = x - \frac{\left(B^2+4\Omega\right)^2}{A^4}t$.

In similar fashion, inserting the values of the parameters arranged in (18) into solution (17), and uniting the solutions (9–13) and simplifying, we acquire the subsequent closed-form wave solutions for $s = 0$ but $r \neq 0$, respectively.

$$u_{16}(\xi) = \alpha_1 + \alpha_2\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$$

$$u_{17}(\xi) = \alpha_1 + \alpha_2\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$$

$$u_{18}(\xi) = \alpha_1 + \alpha_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$$

$$u_{19}(\xi) = \alpha_1 + \alpha_2\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2.$$

If $s = 0$, then from (13) we admit to a constant (trivial) solution which has no physical use and thus, we did not recorded the solution here.

For Case 2

Proceeding as before, making use of the values of the parameters sort out in (19) into solution formula (17) along with solutions (9–13), we gain the following closed-form solitary wave solutions for $r = 0$ but $s \neq 0$, respectively.

$$u_{21}(\xi) = \beta_1 + \beta_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u_{22}(\xi) = \beta_1 + \beta_2\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u_{23}(\xi) = \beta_1 + \beta_2\left[d + \tfrac{B}{2\psi} + \tfrac{\sqrt{\Delta}}{2\psi}\coth\left(\tfrac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \tfrac{B}{2\psi} + \tfrac{\sqrt{\Delta}}{2\psi}\coth\left(\tfrac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$u_{24}(\xi) = \beta_1 + \beta_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$u_{25}(\xi) = \beta_1 + \beta_2\left[d + \frac{B}{2\psi} + \frac{1}{\xi}\right]^{-1} + \beta_3\left[d + \frac{B}{2\psi} + \frac{1}{\xi}\right]^{-2},$$

where $\xi = x - \frac{(B^2+4\Omega)^2}{A^4}t$, $\beta_1 = -\frac{d^2\psi^2+(Bd-E)\psi}{A^2}$, $\beta_2 = \frac{2d^3\psi^2+3Bd^2\psi-2d\Omega+B(Bd-E)}{A^2}$, $\beta_3 = -\frac{d^4\psi^2+2Bd^3\psi-2Ed\Omega+(Bd-E)^2}{A^2}$.

Moreover, plugging in (19) into solution (17) and using the solutions (9–13), respectively, we attain the traveling wave solutions as for $s = 0$ but $r \neq 0$.

$$u_{26}(\xi) = \beta_1 + \beta_2\left[d + \tfrac{\sqrt{\Omega}}{\psi}\tanh\left(\tfrac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \tfrac{\sqrt{\Omega}}{\psi}\tanh\left(\tfrac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u_{27}(\xi) = \beta_1 + \beta_2\left[d - \tfrac{\sqrt{-\Omega}}{\psi}\tan\left(\tfrac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \beta_3\left[d - \tfrac{\sqrt{-\Omega}}{\psi}\tan\left(\tfrac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$u_{28}(\xi) = \beta_1 + \beta_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$
$u_{29}(\xi) = \beta_1 + \beta_2\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2}.$

If $s = 0$, then from (13) we admit to a constant solution and since constant is not physically accessible, we did not recorded the solution here.

For Case 3

Also, for case 3, placing the values of constants provided in (20) into solution formula (17) accompanied with (9–13) and after simplification, respectively, we find the following traveling solutions for $r = 0$ but $s \neq 0$:

$u_{31}(\xi) = \gamma_1 + \gamma_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$
$u_{32}(\xi) = \gamma_1 + \gamma_2\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$
$u_{33}(\xi) = \gamma_1 + \gamma_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$
$u_{34}(\xi) = \gamma_1 + \gamma_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2$
,
$u_{35}(\xi) = \gamma_1 + \gamma_2\left[d + \frac{\mathrm{B}}{2\psi} + \frac{1}{\xi}\right] - \frac{\psi^2}{A^2}\left[d + \frac{\mathrm{B}}{2\psi} + \frac{1}{\xi}\right]^2$, W h e r e $\xi = x - \frac{\left(11 \pm \sqrt{105}\right)\left(B^2 + 4\Omega\right)^2}{8A^4}t$, $\gamma_1 = \frac{-d^2\psi^2 - Bd\psi + \frac{1}{2}\left(1 \pm \frac{\sqrt{105}}{15}\right)\Omega - \frac{1}{8}\left(1 \pm \frac{\sqrt{105}}{15}\right)B^2}{A^2}$, $\gamma_2 = \frac{2d\psi^2 + B\psi}{A^2}$.

Furthermore, substituting (20) into solution (17) along with (9–13) and simplifying, respectively, we find the following traveling solutions for $s = 0$ but $r \neq 0$.

$u_{36}(\xi) = \gamma_1 + \gamma_2\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$
$u_{37}(\xi) = \gamma_1 + \gamma_2\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$
$u_{38}(\xi) = \gamma_1 + \gamma_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$
$u_{39}(\xi) = \gamma_1 + \gamma_2\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right] - \frac{\psi^2}{A^2}\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2$

.When we combine (13) with solution (17), for $s = 0$ yields constant solution, therefore this solution has not been written here.

Similarly, cases 3–5 exert closed-form wave solutions to the CDG equation and for the sake of simplicity those solutions are not reported here.

3.2 The Lax equation

In this sub-section, we will study the nonlocal nonlinear fifth-order Lax Eq. (2). The fifth-order Lax equation is of the form [32]:

$$u_t + \frac{\partial}{\partial x}\left(u_{xxxx} + 10uu_{xx} + 5\left(u_x\right)^2 + 10u^3\right) = 0 \tag{24}$$

The wave variable $\xi = x - ct$ renders Eq. (24) into the following ODE for $u(x, t) = v(\xi)$:

$$-cv' + \left(v^{(iv)} + 10vv'' + 5\left(v'\right)^2 + 10v^3\right)' = 0 \tag{25}$$

Integrating (25) and setting the integration constant to zero, we obtain

$$-cv + v^{(iv)} + 10vv'' + 5\left(v'\right)^2 + 10v^3 = 0 \tag{26}$$

Balancing the highest order linear term $v^{(iv)}$. and nonlinear term of the highest order v^3 in Eq. (26), yields $N=2$. Therefore, the solution shape of the Lax equation is identical to the solution shape (17) and thus has not been reiterated here.

Embedding (17) accompanied with (7) and (8) into (26), the left-hand side is converted into a polynomial in $(d+Y)^N$, $(N=0,1,2,\ldots)$. and $(d+Y)^{-N}$, $(N=1,2,\ldots)$.. We draw together each coefficient of this resulted polynomial and setting them to zero yields an over-determined set of algebraic equations (for simplicity, the equations are not presented here) for $a_0, a_1, a_2, b_1, b_2, c$ and d. Solving these algebraic equations with the help of symbolic computation software, such as, Maple 17, we obtain the following.

Set 1

$$\begin{aligned} a_0 &= -\frac{2\left(d^2\psi^2 + (Bd-E)\psi\right)}{A^2},\ a_1 = 0,\ a_2 = 0, \\ b_1 &= \frac{2\left(d\psi\left(2d^2\psi + 3Bd - 2E\right) + B(Bd-E)\right)}{A^2},\ c = \frac{\left(B^2+4\Omega\right)^2}{A^2}, \\ b_2 &= -\frac{2\left(d\psi\left(d^3\psi + 2Bd^2 - 2Ed\right) + (Bd-E)^2\right)}{A^2} \end{aligned} \tag{27}$$

where $\psi = A - C, \Omega = E\psi, d, A, B, C$ and E are free parameters.

Set 2

$$\begin{aligned} a_0 &= \frac{-2d\psi(d\psi + B) + \left(1 \pm \frac{1}{\sqrt{5}}\right)\Omega - \frac{1}{4}\left(1 \pm \frac{1}{\sqrt{5}}\right)B^2}{A^2},\ a_1 = 0,\ a_2 = 0, \\ b_1 &= \frac{2\left(d\psi\left(2d^2\psi + 3Bd - 2E\right) + B(Bd-E)\right)}{A^2}, \\ b_2 &= -\frac{2\left(d\psi\left(d^3\psi + 2Bd^2 - 2Ed\right) + (Bd-E)^2\right)}{A^2}, \\ c &= \frac{(3 \pm \sqrt{5})\left(B^2+4\Omega\right)^2}{4A^4} \end{aligned} \tag{28}$$

where $\psi = A - C, \Omega = E\psi, d, A, B, C$ and E are free parameters.

Set 3

$$a_0 = \frac{-2d\psi(d\psi + B) + \left(1 \pm \frac{1}{\sqrt{5}}\right)\Omega - \frac{1}{4}\left(1 \pm \frac{1}{\sqrt{5}}\right)B^2}{A^2},\ a_1 = \frac{2\left(2d\psi^2 + B\psi\right)}{A^2},\ a_2 = -\frac{2\psi^2}{A^2},$$
$$b_1 = 0,\ b_2 = 0,\ c = \frac{(3+\sqrt{5})\left(B^2 + 4\Omega\right)^2}{4A^4} \tag{29}$$

where $\psi = A - C$,$\Omega = E\psi$, d, A, B, C and E are free parameters.

Set 4

$$a_0 = -\frac{2\left(d^2\psi^2 + (Bd - E)\psi\right)}{A^2},\ a_1 = \frac{2\left(2d\psi^2 + B\psi\right)}{A^2},\ a_2 = -\frac{2\psi^2}{A^2}$$
$$b_1 = 0,\ b_2 = 0,\ c = \frac{\left(B^2 + 4\Omega\right)^2}{A^4} \tag{30}$$

where $\psi = A - C$,$\Omega = E\psi$, d, A, B, C and E are free parameters.

Set 5

$$a_0 = \frac{\left(B^2 + 4\Omega\right)}{A^2},\ a_1 = 0,\ a_2 = -\frac{2\psi^2}{A^2},\ b_1 = 0,\ b_2 = -\frac{\left(B^2 + 4\Omega\right)^2}{8A^2\psi^2}$$
$$c = \frac{16\left(B^2 + 4\Omega\right)^2}{A^4},\ d = -\frac{B}{2\psi} \tag{31}$$

where $\psi = A - C$,$\Omega = E\psi$, A, B, C and E are free parameters.

Set 6

$$a_1 = 0,\ a_2 = -\frac{2\psi^2}{A^2},\ b_1 = 0,\ b_2 = -\frac{\left(B^2 + 4\Omega\right)^2}{8A^2\psi^2},$$
$$c = \frac{4(3-\sqrt{5})\left(B^2 + 4\Omega\right)^2}{A^4},\ d = -\frac{B}{2\psi} \tag{32}$$

where $\psi = A - C$,$\Omega = E\psi$, A, B, C and E are free parameters.

For Set 1

By using the values of the parameters from set 1 into solution (17) and combining with solutions (9–13), we obtain the following traveling wave solutions for $r = 0$ but $s \neq 0$, respectively.

$$v_{11}(\xi) = \sigma_1 + \sigma_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$
$$v_{12}(\xi) = \sigma_1 + \sigma_2\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$
$$v_{13}(\xi) = \sigma_1 + \sigma_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$
$$v_{14}(\xi) = \sigma_1 + \sigma_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$v_{15}(\xi)=\sigma_1+\sigma_2\left[d+\frac{B}{2\psi}+\frac{1}{\xi}\right]^{-1}+\sigma_3\left[d+\frac{B}{2\psi}+\frac{1}{\xi}\right]^{-2}$,wherein $\xi=x-\frac{\left(B^2+4\Omega\right)^2}{A^4}t$, $\sigma_1=-\frac{2\left(d^2\psi^2+(Bd-E)\psi\right)}{A^2}$, $\sigma_2=\frac{2\left(d\psi\left(2d^2\psi+3Bd-2E\right)+B(Bd-E)\right)}{A^2}$, $\sigma_3=-\frac{2\left(d\psi\left(d^3\psi+2Bd^2-2Ed\right)+(Bd-E)^2\right)}{A^2}$.

Again, by making use of the values of the constants arranged in (27) into solution (17), as well as solutions (9–13) and simplifying, we attain following traveling wave solutions for $s=0$ but $r\neq 0$, respectively.

$$v_{16}(\xi)=\sigma_1+\sigma_2\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1}+\sigma_3\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v_{17}(\xi)=\sigma_1+\sigma_2\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1}+\sigma_3\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v_{18}(\xi)=\sigma_1+\sigma_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1}+\sigma_3\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$v_{19}(\xi)=\sigma_1+\sigma_2\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1}+\sigma_3\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2}$.When we combine (27) with solution (17), for $s=0$ yields constant solution, therefore this solution has not been written here.

For Set 2

In similar fashion, by the determined values of the constants, presenting in set 2, putting into (17) accompanied with (9–13), respectively, we obtain the traveling wave solutions for $r=0$ but $s\neq 0$ as follows:

$$v_{21}(\xi)=\rho_1+\rho_2\left[d+\frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1}+\rho_3\left[d+\frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v_{22}(\xi)=\rho_1+\rho_2\left[d+\frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1}+\rho_3\left[d+\frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v_{23}(\xi)=\rho_1+\rho_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1}+\rho_3\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$v_{24}(\xi)=\rho_1+\rho_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1}+\rho_3\left[d+\frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$v_{25}(\xi)=\rho_1+\rho_2\left[d+\frac{B}{2\psi}+\frac{1}{\xi}\right]^{-1}+\rho_3\left[d+\frac{B}{2\psi}+\frac{1}{\xi}\right]^{-2}$, where $\rho_1=\frac{-2d\psi(d\psi+B)+\left(1\pm\frac{1}{\sqrt{5}}\right)\Omega-\frac{1}{4}\left(1\pm\frac{1}{\sqrt{5}}\right)B^2}{A^2}$, $\rho_2=\frac{2\left(d\psi\left(2d^2\psi+3Bd-2E\right)+B(Bd-E)\right)}{A^2}$, $\rho_3=-\frac{2\left(d\psi\left(d^3\psi+2Bd^2-2Ed\right)+(Bd-E)^2\right)}{A^2}$, $\xi=x-\frac{(3\pm\sqrt{5})\left(B^2+4\Omega\right)^2}{4A^4}t$.

Again setting (28) into solution (17) along with (9–13) and simplifying, we get following traveling wave solutions for $s=0$ but $r\neq 0$, respectively.

$$v_{26}(\xi)=\rho_1+\rho_2\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1}+\rho_3\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v_{27}(\xi)=\rho_1+\rho_2\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1}+\rho_3\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v_{28}(\xi)=\rho_1+\rho_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1}+\rho_3\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$v_{29}(\xi)=\rho_1+\rho_2\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1}+\rho_3\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2}.$$

In the case when $s=0$, from (13) we attain constant solution and since steady is not physically usable, this solution has not been documented here.

For Set 3

By setting the values of the parameters organized in (29) into (27), together with solutions (9–13) and simplifying, we attained the traveling wave solutions as follows for $r=0$ but $s\neq 0$, respectively.

$$v_{31}(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$$

$$v_{32}(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$$

$$v_{33}(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$$

$$v_{34}(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2,$$

$v_{35}(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{B}{2\psi} + \frac{1}{\xi}\right] - \frac{2\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{1}{\xi}\right]^2$, where $\varepsilon_1 = \frac{-2d\psi(d\psi+B)+\left(1-\frac{1}{\sqrt{5}}\right)\Omega-\frac{1}{4}\left(1+\frac{1}{\sqrt{5}}\right)B^2}{A^2}$, $\varepsilon_2 = \frac{2\left(2d\psi^2+B\psi\right)}{A^2}$ and $\xi = x - \frac{(3+\sqrt{5})\left(B^2+4\Omega\right)^2}{4A^4}t$.

We develop the ensuing closed-form solitary wave solutions for $s = 0$ but $r \neq 0$, respectively, by inserting (29) into (27) and using solutions from (9–13) and after simplification:

$$v_{36}(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$$

$$v_{37}(\xi) = \varepsilon_1 + \varepsilon_2\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$$

$$v_{38}(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$$

$$v_{39}(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2.$$

As (13) and (17) are combined, a constant solution for $s = 0$ is sought, but this solution is not written here, since it has no physical significance.

In a similar way, the remaining sets of parameter values contribute closed-form wave solutions of the Lax equation, but these are not presented here to elude pestering.

4 Comparison

Wazwaz examined the CDG equation through several methods, namely, the tanh function method [17, 34, 35], the extended tanh method [36], the tanh-coth method [37], and the sine–cosine method [35]. Since the tanh-coth function method is the generalization of the tanh function method and the extended tanh function method, consequently all the solutions attained by the procedures mentioned above can also be found

Table 1 Comparison of the obtained solutions with the solutions found by Wazwaz in [37]

Solutions found in Wazwaz [37]	Solutions found in this article
$u_1(x,t)=\mu^2\text{sech}^2\left(\mu\left(x-16\mu^4t\right)\right)$	$u_{15}(x,t)=\mu^2\text{sech}^2\left(\delta\left(x-16\mu^4t\right)\right)$ when $d=0$, $B=0$, $\mu=\sqrt{E(A-C)}/A$, $\delta=\sqrt{E/A-C}$
$u_2(x,t)=\mu^2\{\alpha-\tanh^2\left(\mu\left(x-\beta\mu^4t\right)\right)\}$ where $\alpha=\left(15\pm\sqrt{105}\right)/30$ and $\beta=22\mp2\sqrt{105}$	$u_{3_5}(x,t)=\frac{1}{A^2}\{\alpha-\gamma\tanh^2\left(\delta\left(x-\beta\mu^4t\right)\right)\}$ where $\mu=\sqrt{E(A-C)}/A$, $\alpha=\left(15\pm\sqrt{105}\right)/30$, $\beta=22\mp2\sqrt{105}$ and $\gamma=E(A-C)$, $\delta=\sqrt{E/(A-C)}$
$u_3(x,t)=\mu^2\text{csch}^2\left(\mu\left(x-16\mu^4t\right)\right)$	$u_{1_1}(x,t)=\mu^2\text{csch}^2\left(\delta\left(x-16\mu^4t\right)\right)$, when $d=0$, $B=0$, $\mu=\sqrt{E(A-C)}/A$, $\delta=\sqrt{E/A-C}$
$u_4(x,t)=\mu^2\{\alpha-\coth^2\left(\mu\left(x-\beta\mu^4t\right)\right)\}$ where $\alpha=\left(15\pm\sqrt{105}\right)/30$ and $\beta=22\mp2\sqrt{105}$	$u_3(x,t)=\frac{1}{A^2}\{\alpha-\gamma\tanh^2\left(\delta\left(x-\beta\mu^4t\right)\right)\}$ where $\mu=\sqrt{E(A-C)}/A$, $\alpha=\left(15\pm\sqrt{105}\right)/30$, $\beta=22\mp2\sqrt{105}$ and $\gamma=E(A-C)$, $\delta=\sqrt{E/(A-C)}$

through the tanh-coth function method. Therefore, we have compared the attained solutions with those found by the tanh-coth function method only (Wazwaz [37]) in the underneath Table 1.

It is observed from Table that the obtained solutions u_{11}, u_{15}, u_{31} and u_{35} are analogous to the solutions u_1 to u_4 found by Wazwaz in [37]. It is also perceptible that we have attained six sets of solutions of the unidentified parameters and for each set of values we ascertain nine solutions. Consequently, we understand that it can be derived a total of fifty-four solutions. But for the sake of brevity, we have recorded only twenty-seven solutions and the rest of the solutions have not been put down. On the other hand, Wazwaz [37] received only four solutions, all of which we derived. Besides, we have attained remaining fifty new general solutions. By setting specific values of the

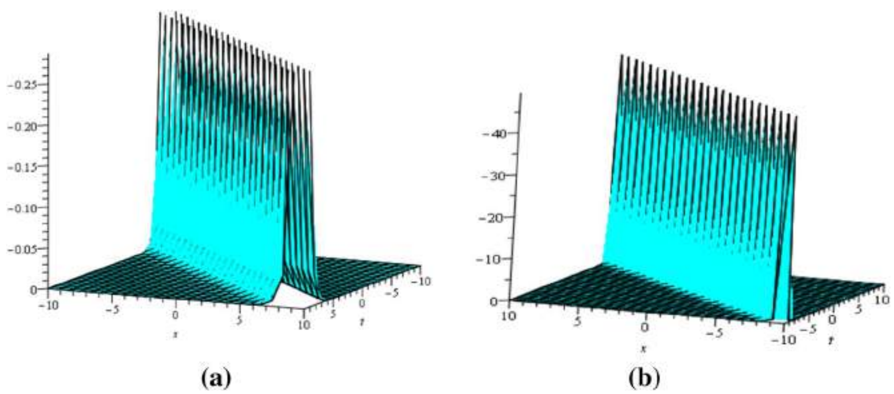

Fig. 1 **a** Is the modulus plot of singular solution $u_{11}(\xi)$ and **b** modulus plot of singular solution v_{11} (ξ)

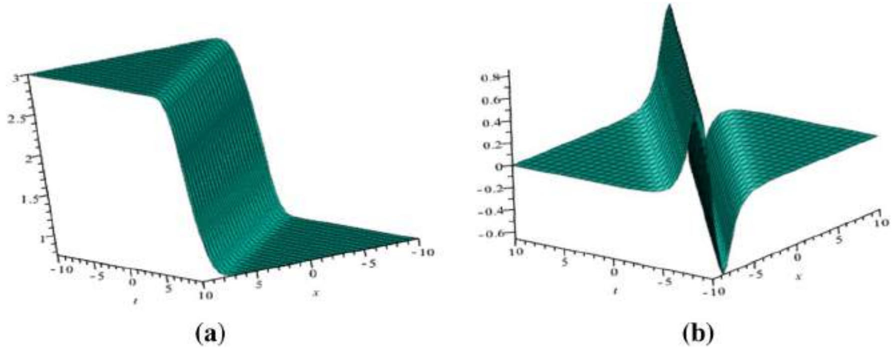

Fig. 2 **a** Is the kink-shaped soliton of solution $v_{18}(\xi)$ and **b** is the singular kink-shaped soliton of $u_{16}(\xi)$

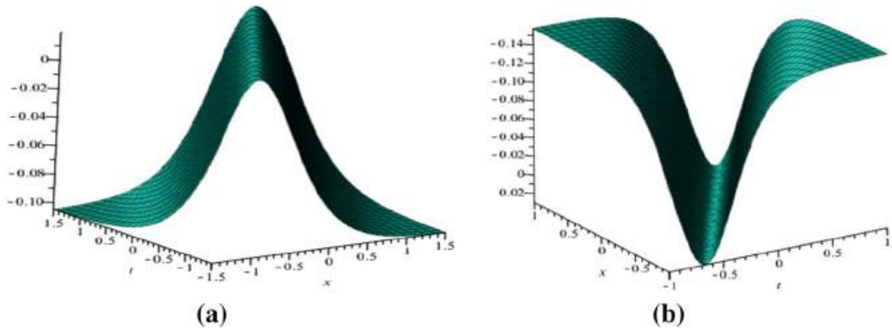

Fig. 3 **a** Is the bell-shaped soliton of solution $u_{33}(\xi)$ and **b** is the anti-bell-shaped soliton solution $u_{36}(\xi)$

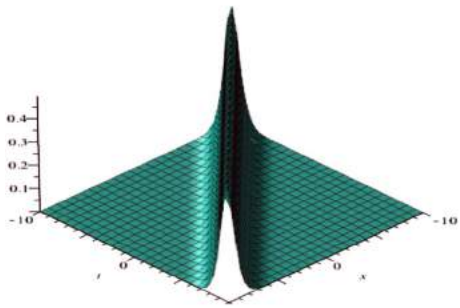

Fig. 4 demonstrates the bell-shaped solitary wave of solution $v_{16}(\xi)$

parameters, we will get many new solutions from these general solutions. That is the specialty of this method.

Analogously, it can be demonstrated that the solutions obtained by means of the new generalized $\left(G'/G\right)$-expansion method transcend all the solutions obtained in the references [17, 34–40]. Also, we might show that all the solutions of the Lax equation found in Ref. [41–47] can be found by the new generalized $\left(G'/G\right)$-expansion method and

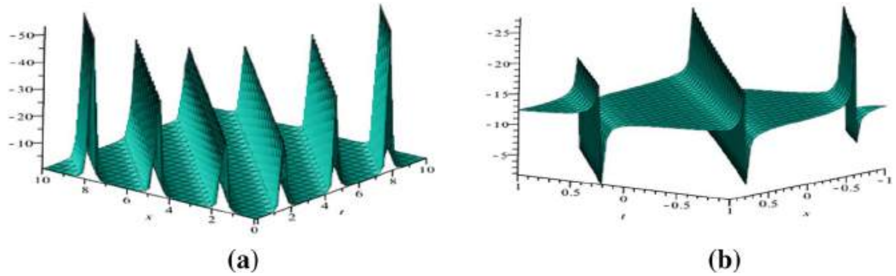

Fig. 5 **a** Shows the periodic wave of solution $u_{34}(\xi)$ and **b** shows periodic wave of solution $v_{32}(\xi)$

some additional solutions will also be found. For simplicity, the comparison Table has not been provided.

5 Graphical representations and physical explanation

Herein, we put forth to represent some three-dimensional figures of the modulus of the extracted solutions of the CDG equation and Lax equation. Figures are constructed by choosing suitable values of the parameters in order to apprehend the internal mechanism of the physical phenomena modulated by Eqs. (1) and (2) with the help of mathematical software Maple 17.

From the obtained solutions, we observe that solutions $u_{11}(\xi), u_{12}(\xi), u_{13}(\xi), u_{14}(\xi)$ and $u_{29}(\xi)$ of the CDG equation and $v_{19}(\xi)$ of the Lax equation are singular solitons. Figure 1a shows the shape of the singular soliton solution $u_{11}(\xi)$ for $d = 1,\ A = 2,\ B = 0,\ C = 1,\ E = 1$ within the interval $-10 \le x, t \le 10$. Moreover, the solutions $u_{21}(\xi)$ and $u_{22}(\xi)$ of the CDG equation and $v_{11}(\xi)$ and $v_{12}(\xi)$ of Lax equation represent soliton solutions. The plot of soliton profile of $v_{11}(\xi)$ for $d = -10,\ A = 2,\ B = 0,\ C = 1,\ E = 1$ within the interval $-10 \le x, t \le 10$ is displayed in Fig. 1b.

Also, solutions $v_{18}(\xi)$, $v_{28}(\xi)$, $v_{31}(\xi)$, $v_{33}(\xi)$, $v_{36}(\xi)$ and $v_{38}(\xi)$ illustrate the kink-shaped soliton solutions. For compactness, only kink-shaped soliton solution $v_{18}(\xi)$ has been plotted and presented in Fig. 2a for the definite values of parameters $d = -1,\ A = 2,\ B = 4,\ C = 3,\ E = 3$ within the range $-10 \le x, t \le 10$. Furthermore, the solution of the CDG equation u_{16} and the solutions of Lax equation $v_{21}(\xi)$, v_{23} and $v_{26}(\xi)$ are singular kink-shaped soliton solutions. For conciseness, only the solution u_{16} has been sketched and displayed in Fig. 2 for $d = 5,\ A = 2,\ B = 0,\ C = 1,\ E = 1$ within the limit $-10 \le x, t \le 10$.

The structure of the achieved solutions $u_{31}(\xi)$, $u_{33}(\xi)$ and $u_{38}(\xi)$ characterized the standard bell-shaped $\left(\text{sech}^2\right)$ soliton and the solution $u_{36}(\xi)$ indicates anti-bell-shaped soliton. In Fig. 3a, we have sketched the bell-shaped soliton of solution $u_{33}(\xi)$ for the specific values of the parameters $d = -1,\ A = 4,\ B = 2,\ C = 3,\ E = 1$ within the interval $-1.5 \le x, t \le 1.5$ and in Fig. 3b the anti-bell-shaped soliton of solution $u_{36}(\xi)$ has been portrayed for $d = -10,\ A = 4,\ B = 0,\ C = 3,\ E = 3$ with range $-1 \le x, t \le 1$.

The solutions $u_{18}(\xi)$, $u_{26}(\xi)$ and $u_{28}(\xi)$ of the CDG equation and the solution $v_{16}(\xi)$ of the Lax equation exhibit the bell-shaped solitary wave. Bell-shaped solitary wave solution profile of $v_{16}(\xi)$ for $d = -10$, $A = 2$, $B = 0$, $C = 1$, $E = 1$ is presented in Fig. 4 within the interval $-10 \le x, t \le 10$.

The solutions $u_{17}(\xi)$, $u_{19}(\xi)$, $u_{22}(\xi)$, $u_{24}(\xi)$, $u_{27}(\xi)$ and $u_{34}(\xi)$ of the CDG equation and $v_{12}(\xi)$, $v_{14}(\xi)$, $v_{17}(\xi)$, $v_{24}(\xi)$, $v_{29}(\xi)$, $v_{34}(\xi)$ and $v_{39}(\xi)$ of the Lax equation projected the periodic traveling wave solutions. The graphical illustration of exact periodic traveling wave solutions of solution $u_{34}(\xi)$ with $d = 1$, $A = 2$, $B = 2$, $C = 3$, $E = 2$ and $-10 \le x, t \le 10$ is presented in Fig. 5a. Again the solutions $v_{22}(\xi)$, $v_{27}(\xi)$, $v_{32}(\xi)$ and $v_{37}(\xi)$ are also periodic traveling wave solutions. Figure 5b illustrates the shape of the periodic traveling wave solution of $v_{32}(\xi)$ for $d = 5$, $A = 2$, $B = 0$, $C = 3$, $E = 5$ within the range $-1 \le x, t \le 1$.

6 Conclusion

In this research work, we succeeded in implementing the new generalized $\left(G'/G\right)$-expansion method to the CDG equation and the Lax equation. We successfully obtained wider class of closed-form solitary wave solutions with a variety of distinct physical structures, such as, soliton, singular soliton, kink, singular kink, bell-shaped soliton, anti-bell-shaped soliton, periodic and bell type solitary wave solutions which are sketched in Figs. 1,2,3,4 and 5 On comparing to our results in this paper with the well-known results obtained in [17, 34–47], most of the obtained solutions are exclusively new. The crucial privilege of this implemented method against other methods is that the method provides more general and huge amount of new wave solutions which validate the superiority of this method. Although this approach has several advantages, it has some limitations also. This approach has not yet been able to analyze the nonlocal equations, such as the nonlocal nonlinear Schrödinger equation, the partially nonlocal Schrödinger equation [48–50]. Therefore, our next project is to work on this method and extend the approach of finding new solutions to these nonlocal nonlinear evolution equations.

Acknowledgements The authors would like to express their sincere thanks to the anonymous referees for their detail comments and valuable suggestions to improve the quality of the article. The authors thank to Taif University Researcher for Supporting project number (TURSP-2020/16), Taif University, Taif, Saudi Arabia.

Declarations

Conflict of interest We certify that no conflict of interest in relation to this article exists.

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